INEQUALITIES FOR SUMS IN NORMED SPACES

MOHAMED ELMURSI

Abstract. In this article, we make analogus to known inequalities for sums for numbers to vectors in normed spaces and prove that satisfies.

Introduction

We use those known numbers p and q that called conjugate exponents which satisfy

$$\frac{1}{p}+\frac{1}{q}=1$$

Lemma 0.1. Cauchy-Schwartz inequality for sums in Normed algebras X

Proof. If $p=q=2$
$\forall x, y \in X$

$$\|xy\| \leq \|x\|\,\|y\|\,.$$

Take two sequences $x=(x_1,x_2,\cdots)$ and $y=(y_1,y_2,\cdots)$ in X, we have $\forall i$

$$0 \leq \|x_i y_i\| \leq \|x_i\|\,\|y_i\|$$

$$\Rightarrow$$

$$\sum_i^{\infty} \|x_i y_i\| \leq \sum_i^{\infty} \|x_i\|\,\|y_i\|$$

$$\leq \left(\sum_i \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_i \|y_i\|^2\right)^{\frac{1}{2}}$$

□

It is difficult to see that the Holder inequality holds:

Lemma 0.2. Holder inequality for sums:

Key words and phrases. Cauchy, Holder, Minkowsky inequality.
This work has been done.

Proof.

$$\sum_i^{\infty} \|x_i y_i\| \leq \sum_i^{\infty} \|x_i\| \, \|y_i\|$$

$$\leq \left(\sum_i \|x_i\|^p \right)^{\frac{1}{p}} \left(\sum_i \|y_i\|^q \right)^{\frac{1}{q}}$$

□

Lemma 0.3. Minkowski inequality for sums with $p = 2$

$$0 \leq \|x_i + y_i\| \leq \|x_i\| + \|y_i\|$$

$$0 \leq \|x_i + y_i\|^2 \leq \{\|x_i\| + \|y_i\|\}^2$$

$$= \|x_i\|^2 + \|y_i\|^2 + 2| \, \|x_i\| \, \|y_i\|$$

$$\Rightarrow$$

$$\sum \|x_i + y_i\|^2 \leq \sum_i \|x_i\|^2 + \sum_i \|y_i\|^2 + 2 \sum_i \|x_i\| \, \|y_i\|$$

$$\leq \sum_i \|x_i\|^2 + 2 \left\{ \left(\sum_i \|x_i\|^2 \right)^{\frac{1}{2}} \left(\sum_i \|y_i\|^2 \right)^{\frac{1}{2}} \right\} + \sum_i \|y_i\|^2$$

$$\Rightarrow$$

$$\left(\sum \|x_i + y_i\|^2 \right)^{\frac{1}{2}} \leq \left(\sum_i \|x_i\|^2 \right)^{\frac{1}{2}} + \left(\sum_i \|y_i\|^2 \right)^{\frac{1}{2}}$$

Lemma 0.4. Minkowski inequality for sums with $(p \geq 1)$

Proof. Let $\|x_i\| = \alpha_i$, $\|y_i\| = \beta \; \forall i$.

$$\|x_i + y_i\| \leq \|x_i\| + \|y_i\| \text{ (Triangle Inequality)}$$

$$\Rightarrow$$

$$\|x_i + y_i\|^p \leq (\|x_i\| + \|y_i\|)^p = |\alpha_i + \beta_i|^p$$

$$\Rightarrow$$

$$\sum_i \|x_i + y_i\|^p \leq \sum_i |\alpha_i + \beta_i|^p$$

$$\Rightarrow$$

$$\left(\sum_i \|x_i + y_i\|^p \right)^{\frac{1}{p}} \leq \left(\sum_i |\alpha_i + \beta_i|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i |\alpha_i|^p \right)^{\frac{1}{p}} + \left(\sum_i |\beta_i|^p \right)^{\frac{1}{p}} \text{ (Minkowski Inequality for numbers)}$$

$$= \left(\sum_i \|x_i\|^p\right)^{\frac{1}{p}} + \left(\sum_i \|y_i\|^p\right)^{\frac{1}{p}}$$

□

Acknowledgments

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Sohag University, Mathematics Institut, Egypt
E-mail address: m_elmursi@yahoo.com